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Abstract. The matrix formulation of automorphisms of an affine Kac–Moody algebra that was developed in a previous paper is here applied to the determination of all the conjugacy classes of involutive automorphisms of the affine Kac–Moody algebra $A_1^{(1)}$, where it is compared with the application of more traditional structural techniques.

1. Introduction

In a previous paper (Cornwell [1], hereafter referred to as 'paper 1') a general matrix formulation of the automorphisms of untwisted affine Kac–Moody algebras was developed. In the present paper this is applied to the special case of $A_1^{(1)}$. This analysis has several objectives. Not only are the results of interest in their own right, but as $A_1^{(1)}$ is the 'simplest' untwisted affine Kac–Moody algebra it provides a perfect example for testing the practical applicability of the concepts of paper 1, and for comparing the matrix method with more traditional structural techniques. In subsequent papers this analysis will be extended first to $A_2^{(1)}$ and then to $A_l^{(1)}$ for values of l greater than two, but it should be noted that $A_1^{(1)}$ has special features that are absent in $A_l^{(1)}$ with $l > 1$, so $A_1^{(1)}$ has to be studied separately from the rest of the $A_l^{(1)}$ family.

The notations and conventions that will be employed in the present paper are those defined in paper 1, with the additional convention that equation labels such as (9) and (1.9) refer to the ninth numbered equation of the present paper and of paper 1 respectively. When the untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}$ is $A_1^{(1)}$, the corresponding simple Lie algebra is A_1, for which the rank l has value 1. The generalized Cartan matrix of $A_1^{(1)}$ is

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{1}$$

The two simple roots of $A_1^{(1)}$ are α_0 and α_1, and as the highest root α_H^0 of A_1 is α_1^0, the relation $\alpha_0 = \delta - \alpha_H$ (where α_H is the extension to $\mathcal{H}$ of α_H^0) implies that the root δ is given by

$$\delta = \alpha_0 + \alpha_1 \tag{2}$$

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and so

$$c = h_\delta = h_{\alpha_0} + h_{\alpha_1}. \tag{3}$$

For these simple roots

$$\langle \alpha_0, \alpha_0 \rangle = \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_1^0, \alpha_1^0 \rangle^0 = \tfrac{1}{2} \tag{4}$$

and

$$\langle \alpha_0, \alpha_1 \rangle = \langle \alpha_1, \alpha_0 \rangle = -\tfrac{1}{2}. \tag{5}$$

Let Γ be the two-dimensional irreducible representation of A_1 in which

$$\Gamma(h^0_{\alpha_1^0}) = \mathbf{h}^0_{\alpha_1^0} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

$$\Gamma(e^0_{\alpha_1^0}) = \mathbf{e}^0_{\alpha_1^0} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{7}$$

$$\Gamma(e^0_{-\alpha_1^0}) = \mathbf{e}^0_{-\alpha_1^0} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

The value of the Dynkin index of this representation is given by $\gamma = \frac{1}{4}$. This representation is equivalent to its contragredient representation, for (1.105) holds with

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

Thus for $A_1^{(1)}$ the set of type 1b involutive automorphisms coincides with the set of type 1a involutive automorphisms and the set of type 2b involutive automorphisms coincides with the set of type 2a involutive automorphisms. Consequently it is sufficient to consider only type 1a and 2a involutive automorphisms. The former set divide into two disjoint subsets with $u = 1$ and $u = -1$, whereas for the determination of the representatives of the conjugacy classes of the latter set it is sufficient to let $u = 1$.

The outline of the present paper is as follows. Section 2 is devoted to the application to $A_1^{(1)}$ of the ideas on Cartan preserving root transformations and Cartan preserving automorphisms described in section 2 of paper 1, the aim being to see how far this approach can be taken in the determination of the conjugacy classes of involutive automorphisms of $A_1^{(1)}$. As will be seen, this approach gives incomplete information, but this deficiency is remedied in the matrix formulation of the succeeding sections. In section 3 the involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$ are analysed, the investigation being extended to involutive automorphisms of type 1a with $u = -1$ in section 4 and to involutive automorphisms of type 2a in section 5. The conclusions are summarized in section 6, and compared there with previous related work.

2. The set of involutive root-preserving transformations of $A_1^{(1)}$

2.1. *Classification of the involutive root-preserving transformations of $A_1^{(1)}$*

When $\tilde{\mathscr{L}} = A_1^{(1)}$, as $l = 1$, (1.8) reduces simply to $\Omega^0 = 2\kappa_1^{\Omega}\Lambda_1^0/\langle\alpha_1^0, \alpha_1^0\rangle^0$, and as

$$\Lambda_1^0 = \tfrac{1}{2}\alpha_1^0 \tag{10}$$

it follows from (4) that

$$\Omega^0 = 2\kappa_1^{\Omega}\alpha_1^0 \tag{11}$$

and

$$\langle\alpha_1^0, \Omega^0\rangle^0 = \kappa_1^{\Omega}. \tag{12}$$

Moreover, the only root preserving transformations τ^0 of the simple Lie algebra $\tilde{\mathscr{L}}^0 = A_1$ are such that

$$\tau^0(\alpha_1^0) = \epsilon\alpha_1^0 \tag{13}$$

where

$$\epsilon = 1 \text{ or } -1. \tag{14}$$

Thus the involutive condition (1.19) is always satisfied, while the involutive condition (1.20) becomes $\kappa_1^{\Omega}(\epsilon + \mu)\alpha_1^0 = 0$, which implies that

$$\kappa_1^{\Omega}(\epsilon + \mu) = 0. \tag{15}$$

As $\mu = 1$ or -1, it follows from (13) that there are only four cases to be considered:

(i) $\epsilon = 1$ and $\mu = 1$: in this case (15) implies that $\kappa_1^{\Omega} = 0$ and hence $\Omega^0 = 0$. Clearly the only root transformation of this type is the *identity* transformation

$$\alpha_1^0 \to \alpha_1^0 \qquad \delta \to \delta \qquad \Lambda_0 \to \Lambda_0. \tag{16}$$

(ii) $\epsilon = -1$ and $\mu = 1$: in this case (15) implies that κ_1^{Ω} can take any integer value.

(iii) $\epsilon = 1$ and $\mu = -1$: in this case (15) implies that κ_1^{Ω} can take any integer value.

(iv) $\epsilon = -1$ and $\mu = -1$: in this case (15) implies that $\kappa_1^{\Omega} = 0$ and hence $\Omega^0 = 0$. Clearly the only root transformation of this type is that associated with the Cartan involution of (1.75) and (1.76), which may be denoted by τ_{Cartan}, and for which

$$\tau_{\text{Cartan}}(\alpha_1) = -\alpha_1 \qquad \tau_{\text{Cartan}}(\delta) = -\delta \qquad \tau_{\text{Cartan}}(\Lambda_0) = -\Lambda_0. \tag{17}$$

Some special cases of involutive root-preserving transformations of types (ii) and (iii) will now be investigated, using the fact that (12) and (13) imply that (1.12) reduces to

$$\tau^0(\alpha_1) = \epsilon\alpha_1^0 - \epsilon\kappa_1^0\delta. \tag{18}$$

First, as the scaled root lattice $Q^{0\vee}$ of $\tilde{\mathcal{L}}^0$ consists of all linear functionals ω^0 defined on $\mathcal{H}^0$ that have the form

$$\omega^0 = \sum_{k=1}^{l} \{2\mu_k^\omega / \langle \alpha_k^0, \alpha_k^0 \rangle^0\} \, \alpha_k^0$$

(where each μ_k^ω (for $k = 1, 2, \ldots, l$) is allowed to take any integer value), and as the fundamental weights Λ_k^0 of $\tilde{\mathcal{L}}^0$ are defined by

$$\Lambda_k^0 = \sum_{j=1}^{l} ((\mathbf{A}^0)^{-1})_{jk} \, \alpha_j^0$$

(for $k = 1, 2, \ldots, l$), it follows that ω^0 can be rewritten as

$$\omega^0 = \sum_{k=1}^{l} \{2\kappa_k^\omega / \langle \alpha_k^0, \alpha_k^0 \rangle^0\} \, \Lambda_k^0$$

where

$$\kappa_k^\omega = \sum_{j=1}^{l} \mu_j^\omega (\mathbf{A}^0)_{jk}$$

(for $k = 1, 2, \ldots, l$). Thus the involutive root-preserving transformation τ is a member of the Weyl group of $A_1^{(1)}$ if and only if

$$\Omega^0 = 2\mu_1^\Omega A_{11} \Lambda_1^0 / \langle \alpha_1^0, \alpha_1^0 \rangle^0$$

where μ_1^Ω is an integer. As $A_{11} = 2$, (10) and (4) imply that $\Omega^0 = 4\mu_1^\Omega \alpha_1^0$, and comparison with (11) then shows that $\kappa_1^\Omega = 2\mu_1^\Omega$. Moreover, by (1.1), as $S(\delta) = \delta$ for every $S \in \mathcal{W}$, it follows that $\mu = 1$. Thus the involutive root-preserving transformation τ is a member of the Weyl group of $A_1^{(1)}$ *if and only if* $\mu = 1$ *and* κ_1^Ω is an *even* integer.

In particular, for the Weyl reflection S_{α_1},

$$S_{\alpha_1}(\alpha_0) = \alpha_0 + 2\alpha_1 \tag{19}$$

and

$$S_{\alpha_1}(\alpha_1) = -\alpha_1 \tag{20}$$

which imply (by (18)) that

$$\epsilon = -1 \qquad \mu = 1 \qquad \kappa_1^\Omega = 0. \tag{21}$$

Similarly, for the Weyl reflection S_{α_0},

$$S_{\alpha_0}(\alpha_0) = -\alpha_0 \tag{22}$$

and

$$S_{\alpha_0}(\alpha_1) = 2\alpha_0 + \alpha_1 = -\alpha_1 + 2\delta \tag{23}$$

so that (18) implies that

$$\epsilon = -1 \qquad \mu = 1 \qquad \kappa_1^{\Omega} = 2. \tag{24}$$

By contrast, for the root transformation ρ corresponding to the Dynkin diagram permutation $\alpha_0 \leftrightarrow \alpha_1$,

$$\rho(\alpha_0) = \alpha_1 \qquad \rho(\alpha_1) = \alpha_0 \tag{25}$$

and thus, by (2), $\rho(\delta) = \delta$ and $\rho(\alpha_1) = \delta - \alpha_1$. Consequently for this Dynkin diagram permutation ρ (18) implies that

$$\epsilon = -1 \qquad \mu = 1 \qquad \kappa_1^{\Omega} = 1. \tag{26}$$

(The fact that κ_1^{Ω} is odd indicates that ρ is not a Weyl group transformation.)

The three foregoing examples have been special cases of the type (ii) root transformations. Examples of type (iii) transformations may be obtained by composing each of them with τ_{Cartan}. In particular

$$(\tau_{\text{Cartan}} \circ S_{\alpha_1})(\alpha_0) = -\alpha_0 - 2\alpha_1 \tag{27}$$

and

$$(\tau_{\text{Cartan}} \circ S_{\alpha_1})(\alpha_1) = \alpha_1 \tag{28}$$

so that for $\tau_{\text{Cartan}} \circ S_{\alpha_1}$

$$\epsilon = 1 \qquad \mu = -1 \qquad \kappa_1^{\Omega} = 0. \tag{29}$$

Similarly

$$(\tau_{\text{Cartan}} \circ S_{\alpha_0})(\alpha_0) = \alpha_0 \tag{30}$$

and

$$(\tau_{\text{Cartan}} \circ S_{\alpha_0})(\alpha_1) = -\alpha_1 - 2\alpha_0 = \alpha_1 - 2\delta \tag{31}$$

so that for $\tau_{\text{Cartan}} \circ S_{\alpha_0}$

$$\epsilon = 1 \qquad \mu = -1 \qquad \kappa_1^{\Omega} = -2. \tag{32}$$

Finally

$$(\tau_{\text{Cartan}} \circ \rho)(\alpha_0) = -\alpha_1 \tag{33}$$

and

$$(\tau_{\text{Cartan}} \circ \rho)(\alpha_1) = -\alpha_0 = \alpha_1 - \delta \tag{34}$$

so that for $\tau_{\text{Cartan}} \circ \rho$

$$\epsilon = 1 \qquad \mu = -1 \qquad \kappa_1^{\Omega} = -1. \tag{35}$$

2.2. The classification of the conjugacy classes of involutive root-preserving transformations of $A_1^{(1)}$

Turning to the classification of the conjugacy classes of the group of involutive root-preserving transformations of $A_1^{(1)}$, (1.16)–(1.18) imply that $\tau_1 = \phi\tau_2\phi^{-1}$ if and only if

$$\epsilon_1 = \epsilon_2 \tag{36}$$

$$\kappa_{11}^{\Omega} - \epsilon'\zeta\kappa_{12}^{\Omega} = \zeta(\mu_2 + 1)\kappa^{\Omega} \tag{37}$$

and

$$\mu_1 = \mu_2. \tag{38}$$

Here is is assumed that $\tau_1 = \{\tau_1^0, \Omega_1, \mu_1\}$, $\tau_2 = \{\tau_2^0, \Omega_2, \mu_2\}$, and $\phi = \{\phi^0, \Phi, \zeta\}$, that $\tau_1^0(\alpha_1^0) = \epsilon_1\alpha_1^0$, $\tau_2^0(\alpha_1^0) = \epsilon_2\alpha_1^0$, $\phi^0(\alpha_1^0) = \epsilon'\alpha_1^0$ (cf (13)), and that $\Omega_1 = 2\kappa_{11}^{\Omega}\alpha_1^0$, $\Omega_2 = 2\kappa_{12}^{\Omega}\alpha_1^0$, and $\Phi = 2\kappa^{\Omega}\alpha_1^0$ (cf (11)).

Clearly (36) and (38) imply that

(1) the identity root transformation of (i) is in a conjugacy class of its own;

(2) the involutive root transformations of (ii) can only be conjugate to other root transformations of (ii);

(3) the involutive root transformations of (iii) can only be conjugate to other root transformations of (iii); and

(4) the root transformation τ_{Cartan} of (iv) is in a conjugacy class of its own.

It remains only to show that the sets (ii) and (iii) each contain two conjugacy classes. (To see this consider (37) for the set (ii). The right-hand side of (37) is always even, so if κ_{12}^{Ω} is even then κ_{11}^{Ω} is even, and if κ_{12}^{Ω} is odd then κ_{11}^{Ω} is odd. Moreover, every even value κ_{11}^{Ω} can be obtained from $\kappa_{12}^{\Omega} = 0$ by an appropriate choice of ζ and κ^{Ω}. Similarly, every odd value κ_{11}^{Ω} can be obtained from $\kappa_{12}^{\Omega} = 1$ by an appropriate choice of ζ and κ^{Ω}. Thus the set (ii) contains *two* conjugacy classes. The same arguments show that the set (iii) contains *two* conjugacy classes.)

These considerations show that the group of involutive root-preserving transformations of $A_1^{(1)}$ contains six conjugacy classes. At least one representative has already been identified for each of these classes.

Each involutive root-preserving transformation of $A_1^{(1)}$ corresponds to one or more involutive automorphisms of $A_1^{(1)}$. *However,* it is possible for two involutive automorphisms associated with involutive root transformations that are not conjugate (as root transformations) to be conjugate within the groups of automorphisms. That is, two Cartan-preserving automorphisms can be conjugate via an automorphism that is not a Cartan-preserving automorphism.

The investigation of this matter will start in the next two sub-sections, but, as will soon become apparent, the full study of the problem requires the matrix formulation. That will be the subject of the section that follows.

2.3. Involutive Cartan-preserving automorphisms of $A_1^{(1)}$

The object of this subsection is merely to list all the involutive automorphisms of $A_1^{(1)}$ of the form $\psi = \psi_\tau \exp(\mathrm{ad}(h'))$ (where $h' \in \mathcal{H}$) which correspond to each τ of the group of involutive root-preserving transformation of $A_1^{(1)}$ that is listed explicitly in the previous subsection. It will be recalled that h' may be assumed to satisfy (1.55), and must be such that (1.59) holds. Also (1.31), (1.29) and (1.39) reduce (on using (11) and (4)) to

$$\begin{aligned}
\psi(h_{\alpha_0}) &= (\mu + \epsilon\mu\kappa_1^{\Omega})h_{\alpha_0} + (\mu + \epsilon\mu\kappa_1^{\Omega} - \epsilon)h_{\alpha_1} \\
\psi(h_{\alpha_1}) &= -\epsilon\mu\kappa_1^{\Omega}h_{\alpha_0} + (\epsilon - \epsilon\mu\kappa_1^{\Omega})h_{\alpha_1} \\
\psi(c) &= \mu c \\
\psi(d) &= \mu d + 2\kappa_1^{\Omega}h_{\alpha_1} - \mu(\kappa_1^{\Omega})^2 c.
\end{aligned} \tag{39}$$

(1) τ is the identity mapping: In this case (1.55) does not imply any restriction on the value of h', and (1.59) implies that there are four distinct involutive automorphisms which correspond to $\exp\{\alpha_0(h')\} = \pm 1$ and $\exp\{\alpha_1(h')\} = \pm 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case ψ is the identity mapping, for which

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\
&\psi(c) = c \qquad \psi(d) = d \\
&\psi(e_{\alpha_0}) = e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{-\alpha_1}.
\end{aligned} \tag{40}$$

(ii) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\
&\psi(c) = c \qquad \psi(d) = d \\
&\psi(e_{\alpha_0}) = e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = -e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = -e_{-\alpha_1}.
\end{aligned} \tag{41}$$

(iii) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\
&\psi(c) = c \qquad \psi(d) = d \\
&\psi(e_{\alpha_0}) = -e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = -e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{-\alpha_1}.
\end{aligned} \tag{42}$$

(iv) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\
&\psi(c) = c \qquad \psi(d) = d \\
&\psi(e_{\alpha_0}) = -e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = -e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = -e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = -e_{-\alpha_1}.
\end{aligned} \tag{43}$$

(2) $\tau = S_{\alpha_1}$: It follows from (20), (19), (21) and (1.37) that the most general $h' \in \mathcal{H}$ that satisfies (1.55) is $h' = \kappa_0(h_{\alpha_0} + h_{\alpha_1}) + \kappa_d d = \kappa_0 c + \kappa_d d$ (where κ_0 and κ_d are arbitrary complex numbers), for which $\alpha_0(h') = \kappa_d$ and $\alpha_1(h') = 0$. Consequently there are only two distinct involutive automorphisms, which correspond to $\exp\{\alpha_0(h')\} = \pm 1$ and $\exp\{\alpha_1(h')\} = 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} + 2h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_1} \\ &\psi(c) = c \qquad \psi(d) = d \\ &\psi(e_{\alpha_0}) = e_{\alpha_0+2\alpha_1} \qquad \psi(e_{-\alpha_0}) = e_{-(\alpha_0+2\alpha_1)} \\ &\psi(e_{\alpha_1}) = e_{-\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{\alpha_1}. \end{aligned} \tag{44}$$

(ii) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} + 2h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_1} \\ &\psi(c) = c \qquad \psi(d) = d \\ &\psi(e_{\alpha_0}) = -e_{\alpha_0+2\alpha_1} \qquad \psi(e_{-\alpha_0}) = -e_{-(\alpha_0+2\alpha_1)} \\ &\psi(e_{\alpha_1}) = e_{-\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{\alpha_1}. \end{aligned} \tag{45}$$

(3) $\tau = S_{\alpha_0}$: It follows from (22), (23), (24) and (1.37) that the most general $h' \in \mathcal{H}$ that satisfies (1.55) is $h' = (\kappa_1 - 2\kappa_d)h_{\alpha_0} + \kappa_1 h_{\alpha_1} + \kappa_d d$ (where κ_1 and κ_d are arbitrary complex numbers), for which $\alpha_0(h') = 0$ and $\alpha_1(h') = \kappa_d$. Consequently there are only two distinct involutive automorphisms, which correspond to $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = \pm 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = 2h_{\alpha_0} + h_{\alpha_1} \\ &\psi(c) = c \qquad \psi(d) = 4h_{\alpha_1} - 4c + d \\ &\psi(e_{\alpha_0}) = e_{-\alpha_0} \qquad \psi(e_{-\alpha_0}) = e_{\alpha_0} \\ &\psi(e_{\alpha_1}) = e_{2\alpha_0+\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{-(2\alpha_0+\alpha_1)}. \end{aligned} \tag{46}$$

(ii) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = 2h_{\alpha_0} + h_{\alpha_1} \\ &\psi(c) = c \qquad \psi(d) = 4h_{\alpha_1} - 4c + d \\ &\psi(e_{\alpha_0}) = e_{-\alpha_0} \qquad \psi(e_{-\alpha_0}) = e_{\alpha_0} \\ &\psi(e_{\alpha_1}) = -e_{2\alpha_0+\alpha_1} \qquad \psi(e_{-\alpha_1}) = -e_{-(2\alpha_0+\alpha_1)}. \end{aligned} \tag{47}$$

(4) $\tau = \rho$: It follows from (25), (26) and (1.37) that the most general $h' \in \mathcal{H}$ that satisfies (1.55) is $h' = \kappa_0(h_{\alpha_0} + h_{\alpha_1}) = \kappa_0 c$ (where κ_0 is an arbitrary complex number), for which $\alpha_0(h') = 0$ and $\alpha_1(h') = 0$. Consequently there is only

one distinct involutive automorphism which corresponds to $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$. This is:

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = h_{\alpha_0} \\ &\psi(c) = c \qquad \psi(d) = 2h_{\alpha_1} - c + d \\ &\psi(e_{\alpha_0}) = e_{\alpha_1} \qquad \psi(e_{-\alpha_0}) = e_{-\alpha_1} \\ &\psi(e_{\alpha_1}) = e_{\alpha_0} \qquad \psi(e_{-\alpha_1}) = e_{-\alpha_0}. \end{aligned} \tag{48}$$

(5) $\tau = \tau_{\text{Cartan}}$: In this case $\psi_\tau(h') = -h'$ for all $h' \in \mathscr{H}$, so the most general $h' \in \mathscr{H}$ that satisfies (1.55) is $h' = 0$. Consequently there is only one distinct involutive automorphism which corresponds to $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$. This is the Cartan involution ϕ_{Cartan}:

$$\begin{aligned} &\phi_{\text{Cartan}}(h_{\alpha_0}) = -h_{\alpha_0} \qquad \phi_{\text{Cartan}}(h_{\alpha_1}) = -h_{\alpha_1} \\ &\phi_{\text{Cartan}}(c) = -c \qquad \phi_{\text{Cartan}}(d) = -d \\ &\phi_{\text{Cartan}}(e_{\alpha_0}) = e_{-\alpha_0} \qquad \phi_{\text{Cartan}}(e_{-\alpha_0}) = e_{\alpha_0} \\ &\phi_{\text{Cartan}}(e_{\alpha_1}) = e_{-\alpha_1} \qquad \phi_{\text{Cartan}}(e_{-\alpha_1}) = e_{\alpha_1}. \end{aligned} \tag{49}$$

(6) $\tau = \tau_{\text{Cartan}} \circ S_{\alpha_1}$: It follows from (27), (28), (29) and (1.37) that the most general $h' \in \mathscr{H}$ that satisfies (1.55) is $h' = \kappa_1 h_{\alpha_1}$ (where κ_1 is an arbitrary complex number), for which $\alpha_0(h') = -\kappa_1/2$ and $\alpha_1(h') = \kappa_1/2$. Consequently there are only two distinct involutive automorphisms, which correspond to $\exp\{\alpha_0(h')\} = \exp\{\alpha_1(h')\} = \pm 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(c) = -c \qquad \psi(d) = -d \\ &\psi(e_{\alpha_0}) = e_{-(\alpha_0+2\alpha_1)} \qquad \psi(e_{-\alpha_0}) = e_{\alpha_0+2\alpha_1} \\ &\psi(e_{\alpha_1}) = e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = e_{-\alpha_1}. \end{aligned} \tag{50}$$

(ii) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = h_{\alpha_1} \\ &\psi(c) = -c \qquad \psi(d) = -d \\ &\psi(e_{\alpha_0}) = -e_{-(\alpha_0+2\alpha_1)} \qquad \psi(e_{-\alpha_0}) = -e_{\alpha_0+2\alpha_1} \\ &\psi(e_{\alpha_1}) = -e_{\alpha_1} \qquad \psi(e_{-\alpha_1}) = -e_{-\alpha_1}. \end{aligned} \tag{51}$$

(7) $\tau = \tau_{\text{Cartan}} \circ S_{\alpha_0}$: It follows from (30), (31), (32) and (1.37) that the most general $h' \in \mathscr{H}$ that satisfies (1.55) is $h' = \kappa_0 h_{\alpha_0}$ (where κ_0 is an arbitrary complex number), for which $\alpha_0(h') = \kappa_0/2$ and $\alpha_1(h') = -\kappa_0/2$. Consequently there are only two distinct involutive automorphisms, which correspond to $\exp\{\alpha_0(h')\} = \exp\{\alpha_1(h')\} = \pm 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = -2h_{\alpha_0} - h_{\alpha_1} \\
&\psi(c) = -c \qquad \psi(d) = -4h_{\alpha_1} + 4c - d \\
&\psi(e_{\alpha_0}) = e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = e_{-(2\alpha_0+\alpha_1)} \qquad \psi(e_{-\alpha_1}) = e_{2\alpha_0+\alpha_1}.
\end{aligned} \tag{52}$$

(ii) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = -2h_{\alpha_0} - h_{\alpha_1} \\
&\psi(c) = -c \qquad \psi(d) = -4h_{\alpha_1} + 4c - d \\
&\psi(e_{\alpha_0}) = -e_{\alpha_0} \qquad \psi(e_{-\alpha_0}) = -e_{-\alpha_0} \\
&\psi(e_{\alpha_1}) = -e_{-(2\alpha_0+\alpha_1)} \qquad \psi(e_{-\alpha_1}) = -e_{2\alpha_0+\alpha_1}.
\end{aligned} \tag{53}$$

(8) $\tau = \tau_{\text{Cartan}} \circ \rho$: It follows from (33), (34), (35) and (1.37) that the most general $h' \in \mathcal{H}$ that satisfies (1.55) is $h' = \kappa_0(h_{\alpha_0} - h_{\alpha_1})$ (where κ_0 is an arbitrary complex number), for which $\alpha_0(h') = \kappa_0$ and $\alpha_1(h') = -\kappa_0$. Consequently there are only 2 distinct involutive automorphisms, which correspond to $\exp\{\alpha_0(h')\} = \exp\{\alpha_1(h')\} = \pm 1$. These are:

(i) $\exp\{\alpha_0(h')\} = 1$ and $\exp\{\alpha_1(h')\} = 1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = -h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_0} \\
&\psi(c) = -c \qquad \psi(d) = -2h_{\alpha_1} + c - d \\
&\psi(e_{\alpha_0}) = e_{-\alpha_1} \qquad \psi(e_{-\alpha_0}) = e_{\alpha_1} \\
&\psi(e_{\alpha_1}) = e_{-\alpha_0} \qquad \psi(e_{-\alpha_1}) = e_{\alpha_0}.
\end{aligned} \tag{54}$$

(ii) $\exp\{\alpha_0(h')\} = -1$ and $\exp\{\alpha_1(h')\} = -1$: In this case

$$\begin{aligned}
&\psi(h_{\alpha_0}) = -h_{\alpha_1} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_0} \\
&\psi(c) = -c \qquad \psi(d) = -2h_{\alpha_1} + 2c - d \\
&\psi(e_{\alpha_0}) = -e_{-\alpha_1} \qquad \psi(e_{-\alpha_0}) = -e_{\alpha_1} \\
&\psi(e_{\alpha_1}) = -e_{-\alpha_0} \qquad \psi(e_{-\alpha_1}) = -e_{\alpha_0}.
\end{aligned} \tag{55}$$

2.4. *Preliminary investigation of the conjugacy classes of involutive automorphisms of* $A_1^{(1)}$

The object of this subsection is to investigate what can be achieved in the study of the conjugacy classes of involutive automorphisms of $A_1^{(1)}$ using structural methods alone. The starting point is the observation by Gantmacher [2] that every inner automorphism of a semi-simple Lie algebra (including those associated with the Weyl reflections) is conjugate to 'chief inner automorphism' of the form $\exp\{\text{ad}(h')\}$ (for some h' of its Cartan subalgebra). However, a chief inner automorphism cannot be conjugate to an automorphism associated with a Weyl reflection via a Cartan preserving automorphism (as in (1.56)), because this requires that the corresponding root transformations must be conjugate, which is impossible as $\exp\{\mathrm{ad}(h')\}$ corresponds to the identity root transformation, which, as noted earlier, is in a class of its own.

The first stage is to determine explicitly the non-Cartan-preserving automorphism that conjugates the involutive automorphism $\psi^0_{\tau^0}$ of the simple complex Lie algebra A_1 corresponding to $\tau^0 = S^0_{\alpha^0_1}$ to a chief inner automorphism of A_1. In fact this non-Cartan-preserving automorphism has been found previously (Cornwell [3, 4]) for other reasons. Denoting it by $V^0_{\alpha^0_1}$, all that is required is to translate the previously quoted form into the conventions of paper 1 (and of Cornwell [5]), in which it becomes

$$V^0_{\alpha^0_1} = \exp\{\mathrm{ad}(ia(e^0_{\alpha^0_1} - e^0_{-\alpha^0_1}))\} \tag{56}$$

where

$$a = \pi/\{8(\alpha^0_1, \alpha^0_1)^0\}^{1/2} = \tfrac{1}{2}\pi. \tag{57}$$

Then

$$\begin{aligned} V^0_{\alpha^0_1}(e^0_{\alpha^0_1} + e^0_{-\alpha^0_1}) &= -2ih^0_{\alpha^0_1} \\ V^0_{\alpha^0_1}(i(e^0_{\alpha^0_1} - e^0_{-\alpha^0_1})) &= i(e^0_{\alpha^0_1} - e^0_{-\alpha^0_1}) \\ V^0_{\alpha^0_1}(ih^0_{\alpha^0_1}) &= \tfrac{1}{2}(e^0_{\alpha^0_1} + e^0_{-\alpha^0_1}). \end{aligned} \tag{58}$$

With the involutive automorphism $\psi^0_{\tau^0}$ of the simple complex Lie algebra A_1 corresponding to $\tau^0 = S^0_{\alpha^0_1}$ being defined by

$$\begin{aligned} \psi^0_{\tau^0}(h^0_{\alpha^0_1}) &= -h^0_{\alpha^0_1} \\ \psi^0_{\tau^0}(e^0_{\alpha^0_1}) &= e^0_{-\alpha^0_1} \\ \psi^0_{\tau^0}(e^0_{-\alpha^0_1}) &= e^0_{\alpha^0_1}. \end{aligned} \tag{59}$$

it is easily checked that by acting with both sides on $h^0_{\alpha^0_1}$, $e^0_{\alpha^0_1}$, and $e^0_{-\alpha^0_1}$ in turn that

$$V^0_{\alpha^0_1} \circ \psi^0_{\tau^0} \circ (V^0_{\alpha^0_1})^{-1} = \exp\{\mathrm{ad}(h^{0'})\} \tag{60}$$

where $h^{0'}$ is such that

$$\exp\{\alpha^0_1(h^{0'})\} = -1 \tag{61}$$

which establishes the conjugacy stated earlier.

The next stage is to extend these ideas to the Kac–Moody algebra $A_1^{(1)}$. Let $\psi^{0\pm}_{\tau^0}$ be the two automorphisms of $A_1^{(1)}$ that are obtained from $\psi^0_{\tau^0}$ by the definitions

$$\psi^{0\pm}_{\tau^0}(t^j \otimes a^0) = (\pm 1)^j t^j \otimes \psi^0_{\tau^0}(a^0) \tag{62}$$

(for all $a^0 \in A_1$ and every integer j),

$$\psi^{0\pm}_{\tau^0}(c) = c \tag{63}$$

and

$$\psi^{0\pm}_{\tau^0}(d) = d. \tag{64}$$

Then $\psi^{0+}_{\tau^0}$ and $\psi^{0-}_{\tau^0}$ are the two involutive automorphisms of $A_1^{(1)}$ that are associated with the Weyl reflection S_{α_1} and which were listed previously as (44) and (45) respectively. Similarly, let $V^{0+}_{\alpha_1^0}$ be the automorphism of $A_1^{(1)}$ that is obtained from $V^0_{\alpha_1^0}$ by the definition

$$V^{0+}_{\alpha_1^0}(t^j \otimes a^0) = t^j \otimes V^0_{\alpha_1^0}(a^0) \tag{65}$$

(for all $a^0 \in A_1$ and every integer j),

$$V^{0+}_{\alpha_1^0}(c) = c \tag{66}$$

and

$$V^{0+}_{\alpha_1^0}(d) = d. \tag{67}$$

Then

$$V^{0+}_{\alpha_1^0} \circ \psi^{0\pm}_{\tau^0} \circ (V^{0+}_{\alpha_1^0})^{-1} = \exp\{\mathrm{ad}(h^{\pm'})\} \tag{68}$$

where $h^{\pm'}$ are elements of $\mathscr{H}$ such that

$$\exp\{\alpha_0(h^{\pm'})\} = -1 \qquad \exp\{\alpha_1(h^{\pm'})\} = \mp 1. \tag{69}$$

This establishes that the automorphisms (44) and (45) associated with the Weyl reflection S_{α_1} are conjugate to the automorphisms (43) and (41) respectively.

The Cartan involution ϕ_{Cartan} of $A_1^{(1)}$ of (1.75), (1.76) and (49) can be treated in a similar way. As

$$\phi_{\mathrm{Cartan}}(t^j \otimes a^0) = t^{-j} \otimes (\phi^0_{\mathrm{Cartan}}(a^0)) \tag{70}$$

(for all $a^0 \in A_1$ and every integer j), where ϕ^0_{Cartan} is the Cartan involution of A_1, which actually coincides with the involutive automorphism $\psi^0_{\tau^0}$ of the simple complex Lie algebra A_1 corresponding to $\tau^0 = S^0_{\alpha_1^0}$ that was defined by (59), it follows from (60) that

$$V^0_{\alpha_1^0} \circ \phi_{\mathrm{Cartan}} \circ (V^0_{\alpha_1^0})^{-1}(t^j \otimes a^0) = t^{-j} \otimes (\exp\{\mathrm{ad}(h^{0'})\}(a^0)) \tag{71}$$

(for all $a^0 \in A_1$ and every integer j). Consequently $V^0_{\alpha_1^0} \circ \phi_{\mathrm{Cartan}} \circ (V^0_{\alpha_1^0})^{-1}$ is identical to the automorphism (51) that is associated with the root transformation $\tau = \tau_{\mathrm{Cartan}} \circ S_{\alpha_1}$, so ϕ_{Cartan} is conjugate to (51).

Because the Weyl reflections S_{α_1} and S_{α_0} are conjugate via root transformation ρ corresponding to the Dynkin diagram permutation $\alpha_0 \leftrightarrow \alpha_1$ of (25), the automorphisms (44) and (45) associated with S_{α_1} are conjugate via the corresponding automorphism (48) to the automorphisms (46) and (47) associated with S_{α_0}. Similarly, the automorphisms (50) and (51) associated with $\tau_{\text{Cartan}} \circ S_{\alpha_1}$ are conjugate via the automorphism (48) to the automorphisms (52) and (53) associated with $\tau_{\text{Cartan}} \circ S_{\alpha_0}$.

These arguments allow one to deduce the following sets of conjugate involutive automorphisms of $A_1^{(1)}$:

(i) (40) (which is certainly in a class of its own as it is the identity);
(ii) (41), (45), (47);
(iii) (42);
(iv) (43), (44), (46);
(v) (48);
(vi) (49), (51), (53);
(vii) (50), (52);
(viii) (54); and
(ix) (55).

However, it remains to be determined whether these sets form *disjoint* classes. As the method just given relies heavily on inspired guesswork, it does not provide a *systematic* means of establishing whether two automorphisms are conjugate. Fortunately the matrix formulation does provide such a systematic treatment, as will be demonstrated in the following sections. Indeed the matrix formulation will show that there are only *seven* conjugacy classes of involutive automorphisms of $A_1^{(1)}$, and the analysis of sections 4 and 5 will demonstrate that the members of the above sets (iii) and (iv) are mutually conjugate, and that the members of these sets (viii) and (ix) are mutually conjugate.

3. Study of the involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$

3.1. Determination of the involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$

As mentioned in section 2 of paper 1, every conjugacy class of involutive automorphisms of $\tilde{\mathcal{L}}$ contains at least one Cartan-preserving involutive automorphism. For $\tilde{\mathcal{L}} = A_1^{(1)}$ each such Cartan-preserving involutive automorphism is associated with a root transformation τ^0 of A_1 such that $\tau^0(\alpha_1^0) = \pm\alpha_1^0$. These two cases will first be considered separately:

3.1.1. Involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$. The most general 2×2 matrix $\mathsf{U}(t)$ that satisfies

$$\mathsf{U}(t)\mathbf{h}^0_{\alpha_1^0}\mathsf{U}(t)^{-1} = \mathbf{h}^0_{\alpha_1^0} \tag{72}$$

with both $\mathsf{U}(t)$ and $\mathsf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by

$$\mathsf{U}(t) = \begin{pmatrix} \eta_1' t^{k_1'} & 0 \\ 0 & \eta_2' t^{k_2'} \end{pmatrix}$$

where η_1' and η_2' are arbitrary non-zero complex numbers and k_1' and k_2' are arbitrary integers. However, (1.111) shows that $(\eta_1')^{-1}t^{-k_1'}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_2 = (\eta_2')/(\eta_1')$ and $k_2 = k_2' - k_1'$, it follows that the most general automorphism of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \eta_2 t^{k_2} \end{pmatrix} \tag{73}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.136) now reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_2 = 0$ and $\eta_2 = \pm 1$. Thus there are *only two* involutive automorphisms of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$, and (by (1.67), (1.71), (1.73)) and (1.141)) these are:

(i) the identity automorphism (40), which corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{74}$$

(ii) the involutive automorphism (43), which corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0. \tag{75}$$

3.1.2. Involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$. The most general 2×2 matrix $\mathbf{U}(t)$ that satisfies

$$\mathbf{U}(t)\, \mathbf{h}^0_{\alpha_1^0} \mathbf{U}(t)^{-1} = -\mathbf{h}^0_{\alpha_1^0} \tag{76}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by

$$\mathbf{U}(t) = \begin{pmatrix} 0 & \eta_1' t^{k_1'} \\ \eta_2' t^{k_2'} & 0 \end{pmatrix}$$

where η_1' and η_2' are arbitrary non-zero complex numbers and k_1' and k_2' are arbitrary integers. However, (1.111) shows that $(\eta_1')^{-1}t^{-k_1'}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_2 = (\eta_2')/(\eta_1')$ and $k_2 = k_2' - k_1'$, it follows that the most general automorphism of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} \tag{77}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.136) again reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, but this now imposes no additional constraints on k_2 and η_2. However the involutive condition (1.138) implies that $\xi = -(k_2)^2$. Thus there is a family of involutive automorphisms of type 1a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$. By (1.67), (1.71), (1.73) and (1.141) these are:

$$\begin{aligned}
\psi(h_{\alpha_0}) &= (1-k_2)h_{\alpha_0} + (2-k_2)h_{\alpha_1} \\
\psi(h_{\alpha_1}) &= k_2 h_{\alpha_0} + (k_2 - 1)h_{\alpha_1} \\
\psi(c) &= c \\
\psi(d) &= 2k_2 h_{\alpha_1} - (k_2)^2 c + d \\
\psi(e_{\alpha_0}) &= -(\eta_2)^{-1} e_{(1-k_2)\alpha_0 + (2-k_2)\alpha_1} \\
\psi(e_{-\alpha_0}) &= -\eta_2 e_{-(1-k_2)\alpha_0 - (2-k_2)\alpha_1} \\
\psi(e_{\alpha_1}) &= -\eta_2 e_{k_2\alpha_0 + (k_2-1)\alpha_1} \\
\psi(e_{-\alpha_1}) &= -(\eta_2)^{-1} e_{-k_2\alpha_0 - (k_2-1)\alpha_1}.
\end{aligned} \tag{78}$$

Comparison with (39) shows that this corresponds to the involutive automorphism considered in the previous section with

$$\epsilon = -1 \qquad \kappa_1^\Omega = k_2 \qquad \mu = 1. \tag{79}$$

Three special cases are worth noting:

(i) The choice $k_2 = 0$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{80}$$

which corresponds (by (78)) to the involutive automorphism (44);

(ii) The choice $k_2 = 2$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ -t^2 & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = -4 \tag{81}$$

which corresponds (by (78) to the involutive automorphism (46);

(iii) The choice $k_2 = 1$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = -1 \tag{82}$$

which corresponds (by (78)) to the involutive automorphism (48).

3.2. Identification of conjugacy classes of involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$

It will now be shown that there are *three* conjugacy classes of involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = 1$. These three classes contain the following Cartan-preserving automorphisms:

(i) the identity automorphism (40), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (74).

(ii) the family of involutive automorphisms (78) corresponding to the type 1a automorphism with

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = -(k_2)^2 \tag{83}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary *even* integer, *together with* the involutive automorphism (43), which corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{84}$$

(iii) the family of involutive automorphisms (78) corresponding to the type 1a automorphism with

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = -(k_2)^2 \tag{85}$$

where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary *odd* integer.

It is clear that the identity automorphism is in a class of its own, so attention will be concentrated on establishing that the other two sets just quoted do indeed form conjugacy classes.

Firstly, as

$$\mathbf{S}(t) \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} (\mathbf{S}(t))^{-1} = \eta t^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{86}$$

with

$$\mathbf{S}(t) = \begin{pmatrix} 1 & (\eta_2)^{-1/2} t^{k_2}/2 \\ 1 & -(\eta_2)^{-1/2} t^{k_2}/2 \end{pmatrix} \tag{87}$$

where

$$\eta = (\eta_2)^{1/2} \qquad k = \tfrac{1}{2} k_2 \tag{88}$$

and as all the entries of the matrix $\mathbf{S}(t)$ of (87) are Laurent polynomials in t if k_2 is even, it follows from (1.158) that if k_2 is even the involutive automorphism (78) corresponding to (83) is conjugate to the involutive automorphism (43) corresponding to (84) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (87) and to $s = 1$.

Turning to the case in which k_2 is odd, it is clear that the argument that has just been given fails, for some of the entries of the matrix $\mathbf{S}(t)$ of (87) are no longer Laurent polynomials in t. However, as

$$\mathbf{S}(t) \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} (\mathbf{S}(t))^{-1} = \eta t^k \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix} \tag{89}$$

with

$$\mathbf{S}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i}(\eta_2)^{-1/2} t^{-(k_2+1)}/2 \end{pmatrix} \tag{90}$$

where

$$\eta = i(\eta_2)^{1/2} \qquad k = \tfrac{1}{2}(k_2 - 1) \tag{91}$$

and as all the entries of the matrix $\mathbf{S}(t)$ of (90) are Laurent polynomials in t if k_2 is odd, it follows from (1.158) that if k_2 is odd the involutive automorphism (78) corresponding to (83) is conjugate to the involutive automorphism (48) corresponding to (82) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (90) and $s = 1$.

It remains only to show if k_2 is an *odd* integer then none of the involutive automorphisms (78) corresponding to (85) can be conjugate via a type 1a or 2a automorphism to the involutive automorphism (43) corresponding to (84). This follows from (1.158) and (1.178), for both of these conjugacy conditions can be cast in a form which would require that the matrix $\mathbf{U}(t)$ of (85) be equivalent through a similarity transformation to a diagonal matrix with diagonal entries that are Laurent polynomials in t. This requires that the matrix $\mathbf{U}(t)$ of (85) must have eigenvalues that are Laurent polynomials in t, which is not possible if k_2 is odd.

4. Study of the involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$

4.1. Determination of the involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$

As in the previous section the Cartan-preserving involutive automorphisms associated with the root transformations τ^0 of A_1 such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_1^0) = -\alpha_1^0$ will first be considered separately:

4.1.1. Involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$. Consideration of the most general 2×2 matrix $\mathbf{U}(t)$ that satisfies (72) again leads to the conclusion that the most general automorphism of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ corresponds to (73), where again η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.136) now reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_2 = 0$ and $\eta_2 = \pm 1$. Thus there are *only two* involutive automorphisms of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$, and (by (1.67), (1.71), (1.73) and (1.138)) these are:

(i) the involutive automorphism (42), which corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = -1 \qquad \xi = 0 \tag{92}$$

(ii) the involutive automorphism (41), which corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad u = -1 \qquad \xi = 0. \tag{93}$$

4.1.2. Involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$. Consideration of the most general 2×2 matrix $\mathbf{U}(t)$ that satisfies (76) again leads to the conclusion that the most general automorphism of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ corresponds to (77), where again η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.136) again reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, but although this imposes no additional constraint η_2 it does require that k_2 must be *even*. The other involutive condition (1.138) again implies that $\xi = -(k_2)^2$. Thus there is a family of involutive automorphisms of type 1a with $u = -1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$. By (1.67), (1.71) and (1.73) and (1.138) these are:

$$
\begin{aligned}
\psi(h_{\alpha_0}) &= (1-k_2)h_{\alpha_0} + (2-k_2)h_{\alpha_1} \\
\psi(h_{\alpha_1}) &= k_2 h_{\alpha_0} + (k_2-1)h_{\alpha_1} \\
\psi(c) &= c \\
\psi(d) &= 2k_2 h_{\alpha_1} - (k_2)^2 c + d \\
\psi(e_{\alpha_0}) &= (\eta_2)^{-1} e_{(1-k_2)\alpha_0+(2-k_2)\alpha_1} \\
\psi(e_{-\alpha_0}) &= \eta_2 e_{-(1-k_2)\alpha_0-(2-k_2)\alpha_1} \\
\psi(e_{\alpha_1}) &= -\eta_2 e_{k_2\alpha_0+(k_2-1)\alpha_1} \\
\psi(e_{-\alpha_1}) &= -(\eta_2)^{-1} e_{-k_2\alpha_0-(k_2-1)\alpha_1}.
\end{aligned}
\tag{94}
$$

Comparison with (39) shows that this again corresponds to the involutive automorphism considered in the section 2 with the identification (79). Two special cases are worth noting:

(i) The choice $k_2 = 0$ and $\eta_2 = -1$ gives

$$
\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{95}
$$

which corresponds (by (94)) to the involutive automorphism (45);

(ii) The choice $k_2 = 2$ and $\eta_2 = 1$ gives

$$
\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ t^2 & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = -4 \tag{96}
$$

which corresponds (by (94)) to the involutive automorphism (47).

4.2. Identification of conjugacy classes of involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$

It will now be shown that there is *only one* conjugacy class of involutive automorphisms of $A_1^{(1)}$ of type 1a with $u = -1$. This contains of all the involutive automorphisms listed in the previous subsection.

This result can be established in two stages. First, with k_2 even (as it must be here), the matrix $\mathbf{S}(t)$ of (87) is such that $\mathbf{S}(t) = \mathbf{S}(-t)$, so

$$
\mathbf{S}(t) \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} (\mathbf{S}(-t))^{-1} = \eta t^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{97}
$$

is satisfied with the matrix $\mathbf{S}(t)$ of (87) and with η and k given by (88). As all the entries of the matrix $\mathbf{S}(t)$ of (87) are Laurent polynomials in t if k_2 is even, it follows from (1.158) that the involutive automorphism (94) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ \eta_2 t^{k_2} & 0 \end{pmatrix} \qquad u = -1 \qquad \xi = -(k_2)^2 \tag{98}$$

(where η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary *even* integer) is conjugate to the involutive automorphism (41) corresponding to (93) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (87) and to $s = 1$.

The second stage is to show that the involutive automorphism (42) corresponding to (92) is conjugate to the involutive automorphism (41) corresponding to (93). As

$$\mathbf{S}(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{S}(-t))^{-1} = \eta t^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{99}$$

with

$$\mathbf{S}(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \tag{100}$$

where

$$\eta = 0 \qquad k = 0 \tag{101}$$

it follows from (1.158) that the involutive automorphism (42) corresponding to (92) is conjugate to the involutive automorphism (41) corresponding to (93) via a type 1a automorphism belonging to the matrix $\mathbf{S}(t)$ of (100) and $s = 1$.

5. Study of the involutive automorphisms of $A_1^{(1)}$ of type 2a (with $u = 1$)

5.1. Determination of the involutive automorphisms of $A_l^{(1)}$ of type 2a (with $u = 1$)

As in the previous two subsections the Cartan-preserving involutive automorphisms associated with the root transformations τ^0 of A_1 such that $\tau^0(\alpha_1^0) = \alpha_1^0$ and $\tau^0(\alpha_1^0) = -\alpha_1^0$ will first be considered separately:

5.1.1. Involutive automorphisms of $A_1^{(1)}$ of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$. Consideration of the most general 2×2 matrix $\mathbf{U}(t)$ that satisfies (72) leads to the conclusion that the most general automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$ again corresponds to (73), where again η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.142) now reduces to $\mathbf{U}(t)\ \mathbf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $\eta_2 = \pm 1$ but that k_2 can take any integer value. The other involutive condition (1.143) implies that $\xi = (k_2)^2$.

Thus there is a family of involutive automorphisms of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = \alpha_1^0$, and (by (1.69), (1.71) and (1.73)) these are:

$$\begin{aligned}
\psi(h_{\alpha_0}) &= (k_2 - 1)h_{\alpha_0} + (k_2 - 2)h_{\alpha_1} \\
\psi(h_{\alpha_1}) &= -k_2 h_{\alpha_0} + (1 - k_2)h_{\alpha_1} \\
\psi(c) &= -c \\
\psi(d) &= -2k_2 h_{\alpha_1} + (k_2)^2 c - d \\
\psi(e_{\alpha_0}) &= \eta_2 e_{(k_2-1)\alpha_0+(k_2-2)\alpha_1} \\
\psi(e_{-\alpha_0}) &= (\eta_2)^{-1} e_{-(k_2-1)\alpha_0-(k_2-2)\alpha_1} \\
\psi(e_{\alpha_1}) &= (\eta_2)^{-1} e_{-k_2\alpha_0+(1-k_2)\alpha_1} \\
\psi(e_{-\alpha_1}) &= \eta_2 e_{k_2\alpha_0-(1-k_2)\alpha_1}.
\end{aligned} \tag{102}$$

Comparison with (39) shows that this corresponds to the involutive automorphism considered in the previous section with

$$\epsilon = 1 \qquad \kappa_1^\Omega = -k_2 \qquad \mu = -1. \tag{103}$$

Six special cases are worth noting:

(i) The choice $k_2 = 0$ and $\eta_2 = 1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{104}$$

which corresponds (by (102)) to the involutive automorphism (50).

(ii) The choice $k_2 = 0$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{105}$$

which corresponds (by (102)) to the involutive automorphism (51).

(iii) The choice $k_2 = 2$ and $\eta_2 = 1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \qquad u = 1 \qquad \xi = 4 \tag{106}$$

which corresponds (by (102)) to the involutive automorphism (52).

(iv) The choice $k_2 = 2$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix} \qquad u = 1 \qquad \xi = 4 \tag{107}$$

which corresponds (by (102)) to the involutive automorphism (53).

(v) The choice $k_2 = 1$ and $\eta_2 = 1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \qquad u = 1 \qquad \xi = 1 \tag{108}$$

which corresponds (by (102)) to the involutive automorphism (54).

(vi) The choice $k_2 = 1$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix} \qquad u = 1 \qquad \xi = 1 \tag{109}$$

which corresponds (by (102)) to the involutive automorphism (55).

5.1.2. Involutive automorphisms of $A_1^{(1)}$ of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$. Consideration of the most general 2×2 matrix $\mathbf{U}(t)$ that satisfies (76) leads to the conclusion that the most general automorphism of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$ corresponds to (77), where again η_2 is an arbitrary non-zero complex number and k_2 is an arbitrary integer. The involutive condition (1.136) again reduces to $\mathbf{U}(t)\,\mathbf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which requires that $k_2 = 0$, but imposes no additional constraint on η_2. The other involutive condition (1.143) implies that $\xi = 0$. Thus there is a family of involutive automorphisms of type 2a with $u = 1$ such that $\tau^0(\alpha_1^0) = -\alpha_1^0$, and (by (1.69), (1.71) and (1.73)) these are:

$$\begin{aligned}
&\psi(h_{\alpha_0}) = -h_{\alpha_0} \qquad \psi(h_{\alpha_1}) = -h_{\alpha_1} \\
&\psi(c) = -c \qquad \psi(d) = -d \\
&\psi(e_{\alpha_0}) = -(\eta_2)^{-1} e_{-\alpha_0} \qquad \psi(e_{-\alpha_0}) = -(\eta_2)^{-1} e_{\alpha_0} \\
&\psi(e_{\alpha_1}) = -\eta_2 e_{-\alpha_1} \qquad \psi(e_{-\alpha_1}) = -(\eta_2)^{-1} e_{\alpha_1}.
\end{aligned} \tag{110}$$

One special case is worth noting:

(i) The choice $k_2 = 0$ and $\eta_2 = -1$ gives

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{C} \qquad u = 1 \qquad \xi = 0 \tag{111}$$

which corresponds (by (102)) to the Cartan involution (49).

5.2. Identification of conjugacy classes of involutive automorphisms of $A_1^{(1)}$ of type 2a

It will now be shown that there are *three* conjugacy classes of involutive automorphisms of $A_1^{(1)}$ of type 2a. These three classes contain the following Cartan-preserving automorphisms:

(i) the family of type 2a involutive automorphisms (102) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \pm t^{k_2} \end{pmatrix} \qquad u = 1 \qquad \xi = (k_2)^2 \tag{112}$$

where k_2 is an arbitrary *odd* integer;

(i) the family of type 2a involutive automorphisms (102) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{k_2} \end{pmatrix} \qquad u = 1 \qquad \xi = (k_2)^2 \tag{113}$$

where k_2 is an arbitrary *even* integer;

(iii) the family of type 2a involutive automorphisms (102) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -t^{k_2} \end{pmatrix} \qquad u = 1 \qquad \xi = (k_2)^2 \tag{114}$$

where k_2 is an arbitrary *even* integer, *together with* the family of type 2a involutive automorphisms (110) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 0 & 1 \\ \eta_2 & 0 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{115}$$

where η_2 is an arbitrary non-zero complex number.

It will first be shown that each of these sets contain mutually conjugate elements, and it will then be shown that they are disjoint.

It is easily checked that with $s = \pm 1$

$$\mathbf{S}(t)\begin{pmatrix}1 & 0\\ 0 & \eta_2(st)^{k_2}\end{pmatrix}(\mathbf{S}(s^{-2}t^{-1}))^{-1} = \begin{pmatrix}1 & 0\\ 0 & \eta_2 s^{k_2}t^{(k_2+2\kappa)}\end{pmatrix} \tag{116}$$

where

$$\mathbf{S}(t) = \begin{pmatrix}1 & 0\\ 0 & t^{\kappa}\end{pmatrix} \tag{117}$$

and where κ is an arbitrary integer. Clearly

(i) if k_2 is *odd* all then every odd power of t can appear in the matrix on the right-hand side of (116) by an appropriate choice of κ, and the sign of $\eta_2 s^{k_2}$ is the same as that of η_2 if s is chosen to have the value 1, but these two quantities have opposite signs if $s = -1$;

(ii) if k_2 is *even* all then every even power of t can appear in the matrix on the right-hand side of (116) by an appropriate choice of κ, but the signs of $\eta_2 s^{k_2}$ and η_2 are the same with $s = \pm 1$.

It then follows from (1.174) and (1.175) that the members of family of type 2a involutive automorphisms (102) corresponding to (112) are mutually conjugate via a type 1a automorphism, as are those of the family of type 2a involutive automorphisms (102) corresponding to (113). The same is true of those of the family of type 2a involutive automorphisms (102) corresponding to (114).

Turning to the family of type 2a involutive automorphisms (110) corresponding to (115), it is easily checked that

$$\mathbf{S}(t)\begin{pmatrix}0 & 1\\ \eta_2 & 0\end{pmatrix}(\mathbf{S}(s^{-2}t^{-1}))^{-1} = \eta t^{k}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{118}$$

where

$$\mathbf{S}(t) = \begin{pmatrix}1 & (\eta_2)^{-1/2}\\ -(\eta_2)^{-1/2} & 1\end{pmatrix} \tag{119}$$

and where

$$\eta = (\eta_2)^{1/2} \qquad k = 0. \tag{120}$$

(Here η_2 is an arbitrary non-zero complex number). It then follows from (1.174) that every member of the family of type 2a involutive automorphisms (110) corresponding to (115) is conjugate via a type 1a automorphism corresponding to the matrix $\mathbf{S}(t)$ of (119) and $s = 1$ to the type 2a involutive automorphisms (102) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad u = 1 \qquad \xi = 0. \tag{121}$$

It remains only to show that the three sets listed previously are indeed *disjoint* conjugacy classes. To establish this it is sufficient to show that the type 2a involutive automorphisms (110) corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = 1 \qquad \xi = 0 \tag{122}$$

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \qquad u = 1 \qquad \xi = 1 \tag{123}$$

and (121) do not form conjugate pairs. Suppose to the contrary that the automorphisms corresponding to (122) and (123) are conjugate via a type 1a or type 2a automorphism belonging to the matrix $\mathbf{S}(t)$ and to $s = 1$. Then (1.175) and (1.179) both imply that $s^2 = 1$ and then (1.174) and (1.178) both require that

$$\mathbf{S}(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{S}(t^{-1}))^{-1} = \eta t^k \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

which with $t = -1$ reduces to

$$\mathbf{S}(-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{S}(-1))^{-1} = \eta(-1)^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which can never be satisfied. Similar arguments can be applied to the pairs $\{(121), (122)\}$ and $\{(121), (123)\}$.

6. Conclusions regarding the matrix formulation of the involutive automorphisms of $A_1^{(1)}$

The analysis of the previous three sections shows that $A_1^{(1)}$ has *seven* conjugacy classes of involutive automorphisms. These are:

(1) the three conjugacy classes of type 1a involutive automorphisms with $u = 1$ listed in subsection 3.2, for which the representatives may be taken to be:

(i) the identity automorphism (40), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (74);

(ii) the involutive automorphism (43), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (75);

(iii) the involutive automorphism (48), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (82);

(2) the one conjugacy class of type 1a involutive automorphisms with $u = -1$ described in section 4.2, for which the representative may be taken to be the involutive automorphism (41), which corresponds to the type 1a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (93);

(3) the three conjugacy classes of type 2a involutive automorphisms listed in section 5.2, for which the representatives may be taken to be:

(i) the involutive automorphism (55), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (109);

(ii) the involutive automorphism (52), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (106);

(iii) the involutive automorphism (53), which corresponds to the type 2a automorphism with $\mathbf{U}(t)$, u, and ξ being given by (107).

At the end of section 2 *nine* sets of involutive automorphisms of $A_1^{(1)}$ were listed. As has just been noted, the matrix formulation shows that there are only *seven* conjugacy classes of involutive automorphisms of $A_1^{(1)}$, and the analysis of sections 4 and 5 demonstrates that the members of the sets (iii) and (iv) of section 2 are mutually conjugate, and that the members of the sets (viii) and (ix) of section 2 are also mutually conjugate.

These results are in agreement with those obtained earlier by Kobayashi [6] for the derived algebra of $A_1^{(1)}$ by another method. The seven conjugacy classes of involutive automorphisms of $A_1^{(1)}$ consist of the identity automorphism and six conjugacy classes of automorphisms of order 2. In Kobayashi's classification the six order 2 automorphism conjugacy class representatives quoted above are (a), (a′), (c), (b″), (b′) and (b) respectively. (Of course, as Kobayashi has only considered the derived algebra of $A_1^{(1)}$, his analysis did not include any discussion of the action of automorphisms on the scaling element d). The tables of Levstein [7] appear to omit three of the conjugacy classes of involutive automorphisms of $A_1^{(1)}$.

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